# Definitions and key facts for section 5.4

Let $V$ and $W$ be vector spaces with bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ respectively. If $T : V \to W$ is a linear transformation between these two vector spaces, then we can define a matrix transformation between the coordinate vectors of each input $[\mathbf{x}]_{\mathcal{B}}$ and its image $[T(\mathbf{x})]_{\mathcal{C}}$.

In particular, the **matrix for $T$ relative to the bases $\mathcal{B}$ and $\mathcal{C}$** is

$$M = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}.$$

In this case, for every $\mathbf{x}$ in $V$ we have

$$[T(\mathbf{x})]_{\mathcal{C}} = M\,[\mathbf{x}]_{\mathcal{B}}.$$

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In the case that $T$ is a linear transformation from $V$ to itself, so $T : V \to V$, we only require the basis $\mathcal{B}$ to form $M$ and we call it **$\mathcal{B}$-matrix for $T$** and use the notation

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix}.$$

Notice that the $\mathcal{B}$-matrix for $T : V \to V$ satisfies

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$$

for all $\mathbf{x}$ in $V$.

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In the case where $T : \mathbb{R}^n \to \mathbb{R}^n$

$$[T]_{\mathcal{E}} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

is the standard matrix of $T$ and for any other nonstandard basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ we have

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \end{bmatrix}$$

as above. Note for all $\mathbf{x}$ in $\mathbb{R}^n$ we have

$$T(\mathbf{x}) = [T(\mathbf{x})]_{\mathcal{E}} = [T]_{\mathcal{E}}\,[\mathbf{x}]_{\mathcal{E}} = [T]_{\mathcal{E}}\,\mathbf{x} \text{ and } [T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$$

**Fact:** If $A = PDP^{-1}$, where $D$ is a diagonalizable $n \times n$ matrix and $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of $P$, then $D$ is the $\mathcal{B}$-matrix for the transforomation $\mathbf{x} \mapsto A\mathbf{x}$